RecAct 7. Problem 8 Partial Fraction Decomposition.

$$I = \int \frac{x}{(x^2+1)(x-5)}\,dx$$

PFD: $\frac{x}{(x^2+1)(x-5)} = \frac{Ax+B}{x^2+1} + \frac{D}{x-5}$ with constants A, B, D. We're using C as the constant of integration.

Then, $I = \int \frac{Ax+B}{x^2+1} + \frac{D}{x-5}\,dx = \int A\left(\frac{x}{x^2+1}\right) + B\left(\frac{1}{x^2+1}\right) + D\left(\frac{1}{x-5}\right)dx$

$= A\left(\frac{1}{2}\right)\ln|x^2+1| + B\arctan(x) + D\ln|x-5| + C$

↑ We need to identify A, B, D.

From PFD: $(x^2+1)(x-5)\left[\frac{x}{(x^2+1)(x-5)}\right] = \left[\frac{Ax+B}{x^2+1} + \frac{D}{x-5}\right](x^2+1)(x-5)$

$x = (Ax+B)(x-5) + (D)(x^2+1)$ ★

Method 1: Elimination.

If $x=5$: ★: $5 = \cancel{(5A+B)}(0) + D(25+1) = 26D$; $D = \frac{5}{26}$;

If $x=0$: ★: $0 = (0+B)(-5) + \frac{5}{26}(0^2+1) = -5B + \frac{5}{26}$; $5B = \frac{5}{26}$; $B = \frac{1}{26}$;

If $x=1$: ★: $1 = (A+\frac{1}{26})(1-5) + \frac{5}{26}(1^2+1) = (-4)(A+\frac{1}{26}) + \frac{10}{26}$;

$(-4)(A+\frac{1}{26}) = 1 - \frac{10}{26} = \frac{26}{26} - \frac{10}{26} = \frac{16}{26}$;

$A + \frac{1}{26} = (-\frac{1}{4})(\frac{16}{26}) = -\frac{4}{26}$; $A = -\frac{4}{26} - \frac{1}{26} = -\frac{5}{26}$;

Method 2. Coefficient Matching.

★: $x = (Ax+B)(x-5) + (D)(x^2+1) = Ax^2 - 5Ax + Bx - 5B + Dx^2 + D$;

$(0)x^2 + (1)x + (0) = (A+D)x^2 + (-5A+B)x + (-5B+D)$

$\Rightarrow \begin{cases} A+D=0 & ① \\ -5A+B=1 & ② \\ -5B+D=0 & ③ \end{cases}$; $5①$: $\cancel{5A} + 5D = 0$; $②$: $\cancel{-5A} + B = 1$ + ; $B + 5D = 1$ ④ ;

④: $B = 1-5D$; ③: $-5B + D = -5(1-5D) + D = -5 + 25D + D = -5 + 26D = 0$; $D = \frac{5}{26}$;

④: $B = 1 - 5(\frac{5}{26}) = 1 - \frac{25}{26} = \frac{1}{26}$; ①: $A = -D = -\frac{5}{26}$;

Therefore: $\boxed{I = -\frac{5}{26}\left(\frac{1}{2}\right)\ln|x^2+1| + \frac{1}{26}\arctan(x) + \frac{5}{26}\ln|x-5| + C}$